Zero-Frequency Hydrodynamic Coefficients of Vertical Axisymmetric Bodies at a Free Surface

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Numerical results are given for the added mass of vertical axisymmetric bodies piercing a free surface when the frequency tends to zero. The numerical scheme employs a finite-element method which is based on the theory of calculus of variations. Specific body geometries treated here are a family of spheroids (oblate and prolate spheroids) and a family of vertical circular cylinders with various slenderness ratios. The case when the free surface plane does not intersect the center of the spheroid is treated. Specifically, the heave and pitch motions are considered, since classical analytic results are available for the sway motion of a spheroid. The present numerical results can be used in connection with ocean buoy or ocean platform motions by assuming that the body geometry is very small compared with the wavelength of surface waves.

I. Introduction

THERE has been a growing interest in the prediction of the motion responses of ocean platforms and buoys. In many cases of practical interest the buoy is very small compared with the incident wavelength in the ocean. Thus, we may approximate the hydrodynamic coefficients of a buoy by those obtained for the zero-frequency limit.

When we restrict the body geometry to be a body of revolution having a generator parallel to the direction of the gravity force, the general three-dimensional problem reduces to a far simpler set of two-dimensional problems. For this specific type of body geometry, the added mass contains only nine nonzero elements in the general 6×6 matrices. Further, there are only four elements to compute out of nine nonzero elements because of symmetry of the added mass. The sway added mass, for the zero-frequency limit of a spheroid, is identical to the classical results for an infinite fluid. However, the heave added mass or pitch moment of inertia for the zero-frequency limit can not be obtained readily from the classical analytic results in an infinite fluid because the equivalent infinite fluid problem, after reflecting the lower half space into the upper half space, is not as easy to solve analytically as the sway problem.

In this paper the heave added mass and the pitch added moment of inertia of three-dimensional vertical axisymmetric bodies in water of infinite depth are computed as the frequency tends to zero. Specifically, the family of oblate and prolate spheroids and the family of vertical circular cylinders with various slenderness ratios are treated. The effect of changes of the water plane (free surface plane) location to off-center positions in the spheroids is investigated.

As a method of numerical computation we used the finite-element method which is based on the classical variational principle. The present numerical method has been applied previously to water wave problems with success by the present author.[1,2,3] However, the present problem is somewhat different from the previous applications of finite-element methods. In the previous study of the infinite water-depth case, the infinite water depth was truncated and a fictitious constant bottom was constructed at a sufficient depth such that the effect of the finite depth was negligibly small in the solution. In the zero-frequency limit it is difficult to truncate the infinite bottom and to construct a fictitious constant bottom at a finite depth. This is because the shallow water effect enters the solution, however deep the truncated depth is, as the wavelength approaches infinity (i.e., zero-frequency limit). In this paper the finite-element method is applied to the formulation which can be obtained readily from the familiar general formulation for arbitrary frequency by taking the zero-frequency limit. In the heave motion problem, the potential is represented as the sum of a source at the origin and another potential. The strength of the source at the origin is determined readily from consideration of the flux from the body, and the newly defined potential is equivalent to a multipole representation.

II. Formulation of the Problem

We assume that the body has no forward translation velocity, the fluid is incompressible and inviscid, and the flow irrotational. It is convenient to introduce a cylindrical (R,y)

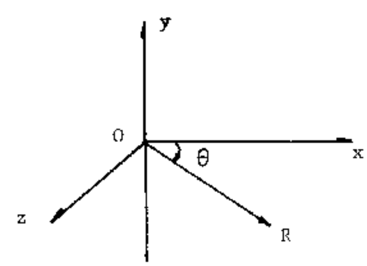

Fig. 1 Coordinate system.

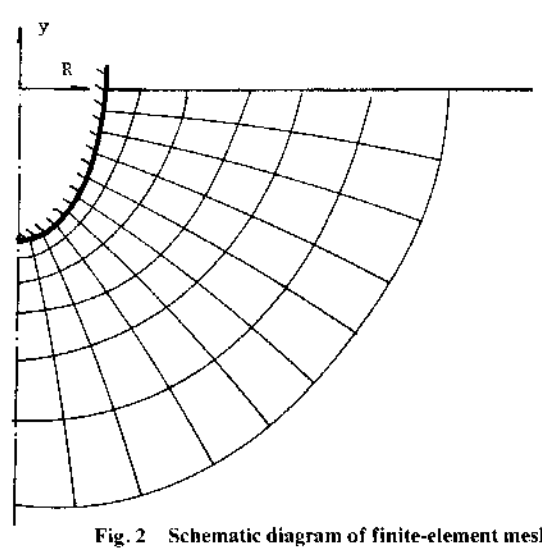

Fig. 2 Schematic diagram of finite-element meshes.

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coordinate system, with the y axis pointing vertically upwards and the $R\theta$ plane in the undisturbed free surface. The R axis coincides with the x axis when θ is zero and with the z axis when θ is $\pi/2$ as shown in Fig. 1, where the xyz coordinate system is rectangular and right-handed.

Since one can find general formulations of this problem in many references (for example Wehausen[4]) we simply will write down the specific formulation for the zero-frequency limit, in terms of the spatial velocity potential ϕ which is real after the time dependence is eliminated

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad (1)$$

in the fluid. As the boundary conditions, we have

$$\partial\phi/\partial y=0 \quad \text{on} \quad y=0 \qquad (2)$$

$$\partial\phi/\partial n=V_n \quad \text{on the body } S_0 \qquad (3)$$

$$\lim_{r\to\infty}\phi=0 \quad , \qquad r=\sqrt{R^2+y^2} \qquad (4)$$

When we consider an axisymmetric body whose axis coincides with the y axis, then the potential ϕ can be assumed to have the form

$$\phi(R,\theta,y)=\sum_{K=0}^{\infty}\varphi^{(K)}(R,y)\cos(K\theta+\beta) \qquad (5)$$

Similarly, the normal velocity on the body can be expressed as

$$V_n=\sum_{K=0}^{\infty}v_n^{(K)}\cos(K\theta+\beta) \qquad (6)$$

Here $\varphi^{(K)}$ and $v_n^{(K)}$ are functions of only R and y, and β is an arbitrary phase angle. Without loss of generality the phase angle β will be taken to be zero here.

In addition we have a new boundary condition along the y axis ($R=0$) due to the reduction of the three-dimensional problem to sets of two-dimensional problems. These are

$$\partial\varphi^{(K)}/\partial n=0 \quad \text{at} \quad R=0 \text{ when } K=0 \qquad (7)$$

and

$$\varphi^{(K)}=0 \quad \text{at} \quad R=0 \text{ when } K\geq 1 \qquad (8)$$

By substituting (5) and (6) into (1) through (4) for $k=0,1,2,\ldots$, we obtain

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\varphi^{(k)}}{\partial R}\right)-\frac{k^2}{R^2}\varphi^{(k)}+\varphi_{yy}^{(k)}=0 \quad (0\leq R,y\leq 0) \qquad (9)$$

$$\varphi_y^{(k)}=0 \quad \text{at} \quad y=0 \qquad (10)$$

$$\varphi_n^{(k)}=v_n^{(k)} \quad \text{on} \quad S_0 \qquad (11)$$

$$\lim_{r\to\infty}\varphi^{(k)}=0 \qquad (12)$$

It should be noted that the reduced problem given in (7-12) is defined only in two dimensions, i.e., in the Ry plane ($R\geq 0$). In the forced motion problem, we consider specifically heave and pitch motions. Then it suffices to solve Eqs. (7-12) for $k=0$ for heave motion, and for $k=1$ for pitch motion. For unit-velocity rigid-body motions, the normal velocity on the body boundary S_0 is expressed as

$$v_n^{(0)}=n_2 \qquad (13)$$

for heave motion, and

$$v_n^{(1)}=r\times n \qquad (14)$$

for pitch motion. Here $\boldsymbol{n}=(n_1,n_2)$ is a unit normal vector directed into the body and $\boldsymbol{r}=(R,y)$ is the position vector on S_0 in the Ry plane.

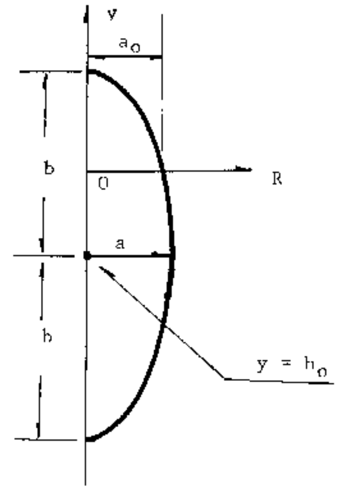

Fig. 3 Floating spheroid.

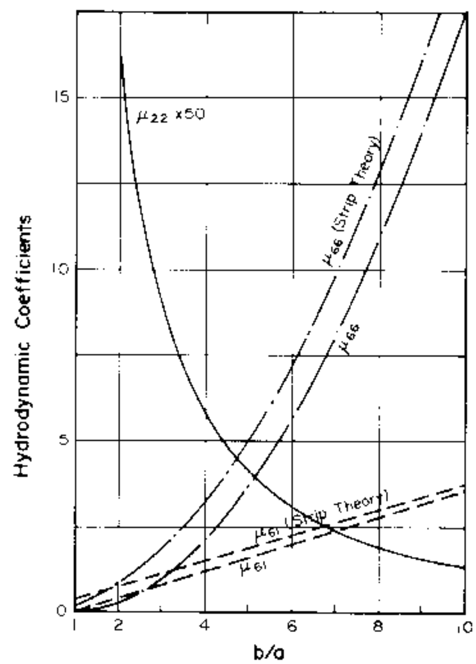

Fig. 4 Hydrodynamic coefficients of prolate spheroids.

Table 1 Hydrodynamic coefficients of spheroids

b/a	μ_{22}	μ_{66}	μ_{61}
0.1	12.840	1.271	−0.299
0.2	5.840	0.549	−0.264
0.3	3.672	0.312	−0.237
0.4	2.621	0.191	−0.208
0.5	2.005	0.117	0.178
0.6	1.604	0.068	−0.144
0.7	1.323	0.036	−0.110
0.8	1.117	0.015	0.075
0.9	0.960	0.004	−0.039
0.95	0.895	0.001	0.021
1.0	0.836	0	0
1.5	0.484	0.073	0.191
2.0	0.323	0.272	0.391
2.51	0.233	0.592	0.600
2.99	0.180	0.993	0.788
3.99	0.116	2.130	1.181
4.99	0.082	3.651	1.565
6.00	0.062	5.790	1.990
7.00	0.049	8.201	2.375
8.00	0.041	11.015	2.767
9.00	0.033	14.235	3.147
10.00	0.028	17.840	3.529

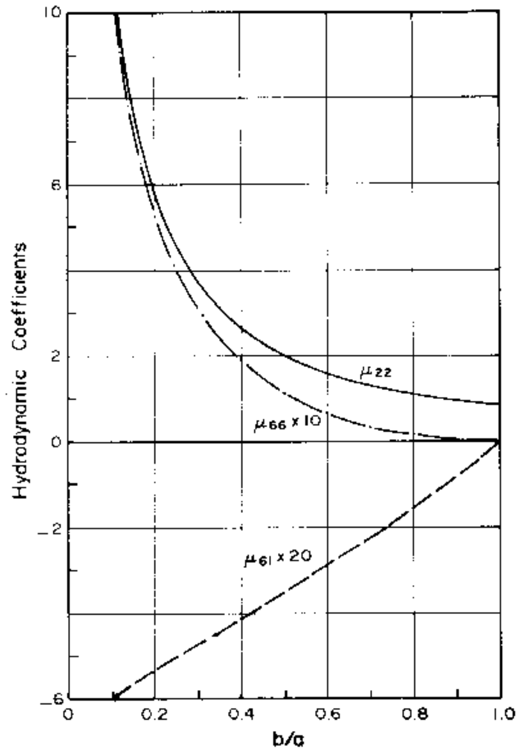

Fig. 5 Hydrodynamic coefficients of oblate spheroids.

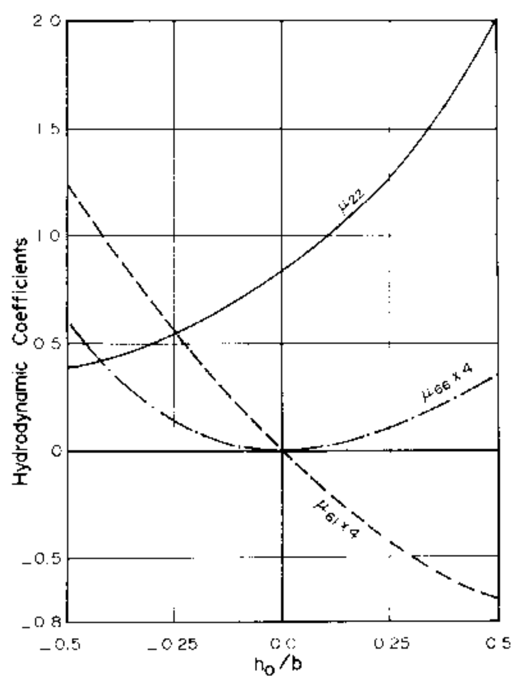

Fig. 6a Effect of change of the waterplane from the center of a sphere.

III. Numerical Procedures

We shall describe a numerical procedure for computing the heave motion. In the heave motion problem of an axisymmetric body piercing the free surface for the zero-frequency limiting case the potential is represented as the sum of two components as follows

$$\varphi^{(0)} = (a_0^2/2)(1/r) + \bar{\varphi}^{(0)} \tag{15}$$

where a_0 is the radius of the body of revolution at the free surface. In heave motion the first term in (15) represents a

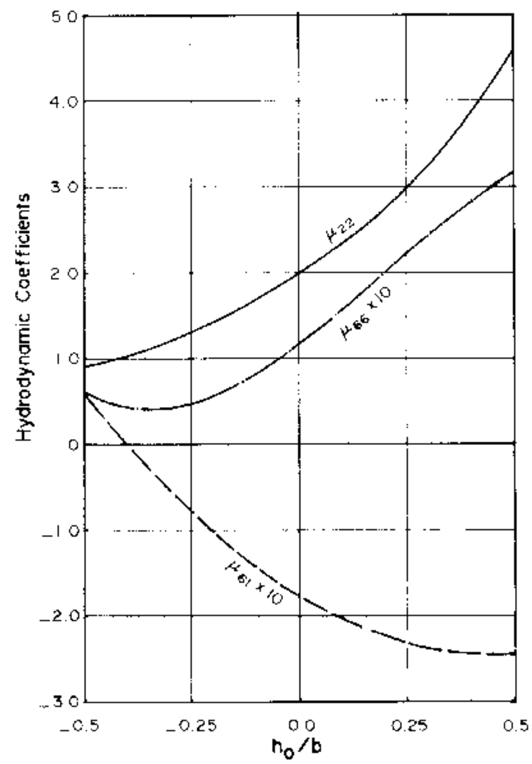

Fig. 6b Effect of change of the centerplane from the center of an oblate spheroid ($a/B=2$).

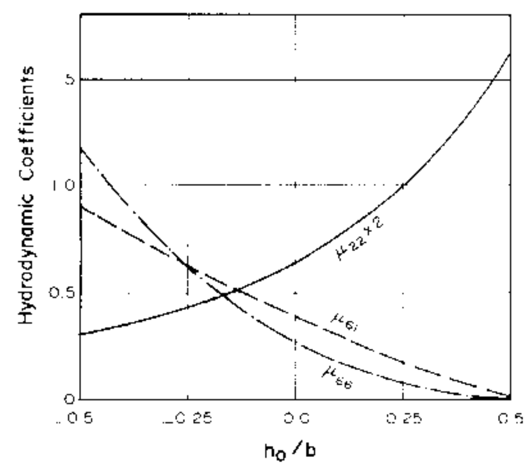

Fig. 6c Effect of change of the waterplane from the center of a prolate spheroid ($b/a=2$).

source of a strength which is determined from the net flux through the body boundary. Then the second term in (15) behaves as a doublet and/or higher-order singularities. Since it is well known that the doublet or any higher-order potential decays more rapidly to zero than the source potential as r increases, it is clear that the numerical computation of $\bar{\varphi}^{(0)}$ is easier than that of $\phi^{(0)}$ without the decomposition in (15). By substituting (15) into (9) through (12), we obtain

$$-\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \bar{\varphi}^{(0)}}{\partial R}\right) + \bar{\varphi}_{yy}^{(0)} = 0 \quad (0 \le R, y \le 0) \tag{16}$$

$$\bar{\varphi}_y^{(0)} = 0 \quad \text{at} \quad y = 0 \tag{17}$$

$$\bar{\varphi}_n^{(0)} = n_2 - (a_0^2/2)(\partial/\partial n)(1/r) \quad \text{on} \quad S_0 \tag{18}$$

$$\lim_{r\to\infty} \bar{\varphi}^{(0)} = 0 \tag{19}$$

In addition, we obtain from (7)

$$\bar{\varphi}_n^{(0)} = 0 \quad \text{at} \quad R = 0 \tag{20}$$

Application of the finite-element method to Eqs. (16-20) is straightforward. The essential condition (19) is imposed at a sufficient distance away from the origin. An example of the finite-element subdivisions in the fluid domain is shown in Fig. 2. The outer boundary replacing the infinite boundary is constructed at a distance approximately 20 times the body length (longer dimension between the radius and the draft of the body).

The numerical procedure for the finite-element method is described elsewhere by Bai.[1,2,3] Throughout the computations an eight-node isoparametric quadrilateral element was used.

It should be mentioned that the infinity boundary condition (12) for $k=1,2,...$ and (19) for $k=0$ can be replaced by the following alternate conditions

$$\lim_{r\to\infty} \varphi_n^{(k)} = 0 \quad \text{for} \quad k=1,2,... \tag{21}$$

$$\lim_{r\to\infty} \bar{\varphi}_n^{(0)} = 0 \tag{22}$$

These alternate infinity conditions were tested in the numerical computations and the agreement with the results obtained from (19) or (12) and those obtained from (21) or (22) was good.

IV. Results and Discussion

Once the desired potentials $\varphi^{(0)}$ for heave and $\varphi^{(1)}$ for pitch are obtained, then the heave added mass $\hat{\mu}_{22}$, the pitch added moment of inertia $\hat{\mu}_{66}$, and the coupling term $\hat{\mu}_{61}$ between sway and pitch motions are computed from

$$\hat{\mu}_{22} = 2\pi\rho \int_{s0} \varphi^{(0)} n_2 R \, ds \tag{23}$$

$$\hat{\mu}_{66} = \pi\rho \int_{s0} R\varphi^{(1)} (\boldsymbol{r} \times \boldsymbol{n}) \, ds \tag{24}$$

$$\hat{\mu}_{61} = \pi\rho \int_{s0} R\, \varphi^{(1)} n_1 \, ds \tag{25}$$

where ρ is the density of the fluid. It is convenient to nondimensionalize the hydrodynamic quantities as follows

$$\mu_{22} = \hat{\mu}_{22}/\rho V \tag{26}$$

$$\mu_{66} = \hat{\mu}_{66}/\rho V a^2 \tag{27}$$

$$\mu_{61} = \hat{\mu}_{61}/\rho V a \tag{28}$$

where a is the radius of a spheroid in the horizontal plane passing through its center or the radius of a vertical circular cylinder and V is the submerged volume of the body.

Floating Spheroid

A family of spheroids as shown in Fig. 3 is treated. The equation of the spheroid family in the Ry plane is given by

$$(R^2/a^2) + [(y-h_0)^2/b^2] = 1 \tag{29}$$

Equation (29) reduces, in limiting cases, to a needle when $a=0$ and $b\neq 0$, and to a disk when $a\neq 0$ and $b=0$.

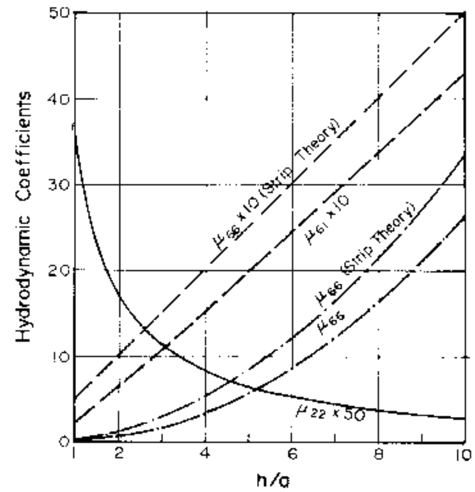

Fig. 7 Hydrodynamic coefficients of a vertical circular cylinder ($h/a > 1$).

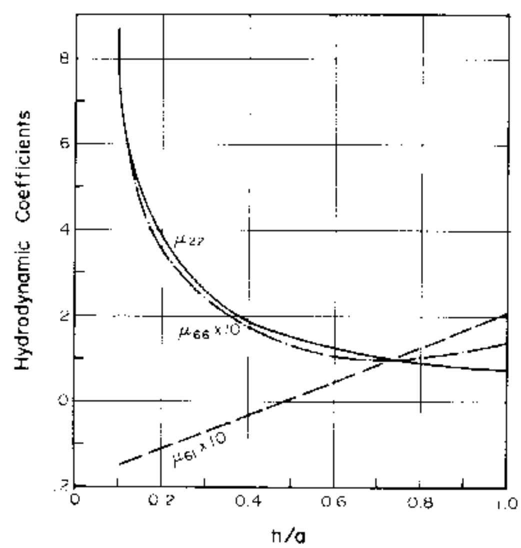

Fig. 8 Hydrodynamic coefficients of a vertical circular cylinder ($h/a < 1$).

Table 2 Effect of change of the waterplane from the center of spheroid

h_0/b	$b/a=0.5$			$b/a=1$			$b/a=2$		
	μ_{22}	μ_{66}	μ_{61}	μ_{22}	μ_{66}	μ_{61}	μ_{22}	μ_{66}	μ_{61}
−0.5	0.915	0.064	0.059	0.392	0.153	0.309	0.157	1.180	0.900
−0.4	1.037	0.045	0.0008	0.439	0.094	0.238	0.173	0.932	0.789
0.3	1.212	0.043	−0.055	0.508	0.051	0.171	0.198	0.720	0.683
−0.2	1.433	0.057	−0.102	0.600	0.021	0.108	0.231	0.541	0.581
−0.1	1.697	0.082	−0.143	0.701	0.005	0.051	0.273	0.393	0.483
0.0	2.005	0.117	−0.177	0.836	0.000	0.000	0.323	0.272	0.391
0.1	2.364	0.157	0.204	0.991	0.005	−0.049	0.385	0.177	0.303
0.2	2.781	0.201	−0.224	1.176	0.018	0.090	0.459	0.106	0.221
0.3	3.274	0.244	−0.238	1.398	0.038	−0.124	0.551	0.056	0.145
0.4	3.867	0.284	0.245	1.671	0.061	−0.152	0.667	0.024	0.075
0.5	4.605	0.318	−0.244	2.018	0.086	−0.171	0.816	0.010	0.013

Table 3 Hydrodynamic coefficients of a circular cylinder for heave and pitch motions

h/a	μ_{22}	μ_{66}	μ_{6l}
0.1	8.725	0.762	−0.150
0.2	3.894	0.353	−0.107
0.4	1.885	0.153	0.031
0.6	1.227	0.104	0.045
0.8	0.904	0.105	0.124
1.0	0.714	0.139	0.205
1.5	0.464	0.332	0.416
2.0	0.340	0.663	0.629
4.0	0.162	3.449	1.526
6.0	0.102	8.585	2.469
8.0	0.075	16.215	3.372
10.0	0.057	26.160	4.292

When $a=b$, (29) gives a sphere. Our results for the computed heave added mass and pitch added moment of inertia were in good agreement with those obtained by Havelock[5] and Kim.[6]

The computed values of μ_{22}, μ_{66}, and μ_{6l} are shown in Fig. 4 for prolate spheroids and in Fig. 5 for oblate spheroids. The numerical values are given in Table 1.

The effect of the depth of submergence of the center of the spheroid is computed for a sphere ($a=b$), and oblate spheroid ($a/b=2$), and a prolate spheroid ($b/a=2$) for 10 off-center submergences. These results are given in Table 2. (When the center of the spheroid is submerged, the value of h_0/b is negative.) These results are shown also in Figs. 6a–c. It is of interest to note that the value of μ_{6l} for $a/b=2$ changes its sign at approximately $h_0/b=-0.4$. It is also of interest to note that the value of μ_{6l} for a sphere has the same sign as μ_{66} when the center of the sphere is submerged ($h_0<0$).

Vertical Circular Cylinder

The computed results for a vertical circular cylinder of radius a and draft h are given in Table 3. These results also are shown in Figs. 7 and 8. It is of interest to note that the sign of μ_{6l} changes at approximately $h/a=0.48$. The value of μ_{6l} varies nearly linearly with h/a as shown in Figs. 7 and 8.

It is also of interest to note in the preceding two types of body geometries that μ_{6l} and μ_{66} can be approximated by the well-known strip theory for very slender bodies as follows:

$$\mu_{6l}=(3/8)\,(b/a) \tag{30}$$

$$\mu_{66}=(1/5)\,(b/a)^2 \tag{31}$$

for a prolate spheroid, and

$$\mu_{6l}=(1/2)\,(h/a) \tag{32}$$

$$\mu_{66}=(3/8)\,(h/a)^2 \tag{33}$$

for a circular cylinder. These simple strip-theory results also are compared with the present numerical results in Figs. 4 and 7.

Acknowledgment

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